

CANONICAL CARTAN CONNECTION FOR 5-DIMENSIONAL CR-MANIFOLDS BELONGING TO GENERAL CLASS III_2

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ABSTRACT

We study the equivalence problem for CR-manifolds belonging to general class III_2 , i.e. the 5-dimensional CR-manifolds of CR-dimension 1 and codimension 3 whose CR-bundle satisfies a degeneracy condition which has been introduced in [9]. For such a CR-manifold M , we construct a canonical Cartan connection on a 6-dimensional principal bundle P on M . This provides a complete set of biholomorphic invariants for M .

1. INTRODUCTION

As highlighted by Henri Poincaré [14] in 1907, the (local) biholomorphic equivalence problem between two submanifolds M and M' of \mathbb{C}^N is to determine whether or not there exists a (local) biholomorphism ϕ of \mathbb{C}^N such that $\phi(M) = M'$. Elie Cartan [2, 3] solved this problem for hypersurfaces $M^3 \subset \mathbb{C}^2$ in 1932, as he constructed a “hyperspherical connection” on such hypersurfaces by using the powerful technique which is now referred to as Cartan’s equivalence method.

Given a manifold M and some geometric data specified on M , which usually appears as a G -structure on M (i.e. a reduction of the bundle of coframes of M), Cartan’s equivalence method seeks to provide a principal bundle P on M together with a coframe ω of 1-forms on P which is adapted to the geometric structure of M in the following sense: an isomorphism between two such geometric structures M and M' lifts to a unique isomorphism between P and P' which sends ω on ω' . The equivalence problem between M and M' is thus reduced to an equivalence problem between $\{e\}$ -structures, which is well understood [10, 15].

We recall that a CR-manifold M is a real manifold endowed with a subbundle L of $\mathbb{C} \otimes TM$ of even rank $2n$ such that

- (1) $L \cap \overline{L} = \{0\}$
- (2) L is formally integrable, i.e. $[L, L] \subset L$.

The integer n is the CR-dimension of M and $k = \dim M - 2n$ is the codimension of M . In a recent attempt [9] to solve the equivalence problem

for CR-manifolds up to dimension 5, it has been shown that one can restrict the study to six different general classes of CR-manifolds of dimension ≤ 5 , which have been referred to as general classes I, II, III₁, III₂, IV₁ and IV₂. The aim of this paper is to provide a solution to the equivalence problem for CR-manifolds which belong to general class III₂, that is the CR-manifolds of dimension 5 and of CR-dimension 1 such that $\mathbb{C} \otimes TM$ is spanned by L, \bar{L} and their Lie brackets up to order no less than 3. More precisely, the following rank conditions hold:

$$3 = \text{rank}_{\mathbb{C}} (L + \bar{L} + [L, \bar{L}]),$$

$$4 = \text{rank}_{\mathbb{C}} (L + \bar{L} + [L, \bar{L}] + [L, [L, \bar{L}]]),$$

$$4 = \text{rank}_{\mathbb{C}} (L + \bar{L} + [L, \bar{L}] + [L, [L, \bar{L}]] + [\bar{L}, [L, \bar{L}]]),$$

$$5 = \text{rank}_{\mathbb{C}} (L + \bar{L} + [L, \bar{L}] + [L, [L, \bar{L}]] + [\bar{L}, [L, \bar{L}]] + [L, [L, [L, \bar{L}]]]),$$

the third one being an exceptional degeneracy assumption.

The main result of the present paper is the following:

Theorem 1. *Let M be a CR-manifold belonging to general class III₂. There exists a 6-dimensional subbundle P of the bundle of coframes $\mathbb{C} \otimes F(M)$ of M and a coframe $\omega := (\Lambda, \tau, \sigma, \rho, \zeta, \bar{\zeta})$ on P such that any CR-diffeomorphism h of M lifts to a bundle isomorphism h^* of P which satisfies $h^*(\omega) = \omega$. Moreover the structure equations of ω on P are of the form:*

$$d\tau = 4\Lambda \wedge \tau + \mathfrak{J}_1 \tau \wedge \zeta - \mathfrak{J}_1 \tau \wedge \bar{\zeta} + 3\mathfrak{J}_1 \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \bar{\zeta},$$

$$d\sigma = 3\Lambda \wedge \sigma$$

$$+ \mathfrak{J}_2 \tau \wedge \rho + \mathfrak{J}_3 \tau \wedge \zeta + \bar{\mathfrak{J}}_3 \tau \wedge \bar{\zeta} + \mathfrak{J}_4 \sigma \wedge \rho \\ - \frac{\mathfrak{J}_1}{2} \sigma \wedge \zeta + \frac{\mathfrak{J}_1}{2} \sigma \wedge \bar{\zeta} + \rho \wedge \zeta + \rho \wedge \bar{\zeta},$$

$$d\rho = 2\Lambda \wedge \rho$$

$$+ \mathfrak{J}_5 \tau \wedge \sigma + \mathfrak{J}_6 \tau \wedge \rho + \mathfrak{J}_7 \tau \wedge \zeta + \bar{\mathfrak{J}}_7 \tau \wedge \bar{\zeta} + \mathfrak{J}_8 \sigma \wedge \rho + \mathfrak{J}_9 \sigma \wedge \zeta \\ + \bar{\mathfrak{J}}_9 \sigma \wedge \bar{\zeta} - \frac{\mathfrak{J}_1}{2} \rho \wedge \zeta + \frac{\mathfrak{J}_1}{2} \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta},$$

$$d\zeta = \Lambda \wedge \zeta$$

$$+ \mathfrak{J}_{10} \tau \wedge \sigma + \mathfrak{J}_{11} \tau \wedge \rho + \mathfrak{J}_{12} \tau \wedge \zeta + \mathfrak{J}_{13} \tau \wedge \bar{\zeta} \\ + \mathfrak{J}_{14} \sigma \wedge \rho + \mathfrak{J}_{15} \sigma \wedge \zeta,$$

$$d\Lambda = \sum_{\nu\mu} X_{\nu\mu} \nu \wedge \mu, \quad \nu, \mu = \tau, \sigma, \rho, \zeta, \bar{\zeta},$$

where $\mathfrak{J}_i, X_{\nu\mu}$, are functions on P .

The model manifold for this class is provided by the CR-manifold $N \subset \mathbb{C}^3$ given by the equations:

$$\begin{aligned} N : \quad w_1 &= \overline{w_1} + 2i z \overline{z}, \\ w_2 &= \overline{w_2} + 2i z \overline{z} (z + \overline{z}), \\ w_3 &= \overline{w_3} + 2i z \overline{z} \left(z^2 + \frac{3}{2} z \overline{z} + \overline{z}^2 \right), \end{aligned}$$

Cartan's equivalence method has been applied to this model in [12], where it has been shown that the coframe $(\Lambda, \tau, \sigma, \rho, \zeta, \overline{\zeta})$ of theorem 1 satisfy the simplified structure equations:

$$\begin{aligned} d\tau &= 4 \Lambda \wedge \tau + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}, \\ d\sigma &= 3 \Lambda \wedge \sigma + \rho \wedge \zeta + \rho \wedge \overline{\zeta}, \\ d\rho &= 2 \Lambda \wedge \rho + i \zeta \wedge \overline{\zeta}, \\ d\zeta &= \Lambda \wedge \zeta, \\ d\overline{\zeta} &= \Lambda \wedge \overline{\zeta}, \\ d\Lambda &= 0, \end{aligned}$$

corresponding to the case where the biholomorphic invariants \mathfrak{J}_i vanish identically. This result, together with the Lie algebra structure of the infinitesimal CR-automorphisms of the model, implies the existence of a Cartan connection on M , which we construct in section 4.

We start in section 2 with the construction of a canonical G -structure P^1 on M , (e.g. a subbundle of the bundle of coframes of M), which encodes the equivalence problem for M under CR-automorphisms in the following sense: a diffeomorphism

$$h : M \longrightarrow M$$

is a CR-automorphism of M if and only if

$$h^* : P^1 \longrightarrow P^1$$

is a G -structure isomorphism of P^1 . We refer to [9, 6, 7] for details on the results summarized in this section and to [15] for an introduction to G -structures. Section 3 is devoted to reduce successively P^1 to four subbundles:

$$P^5 \subset P^4 \subset P^3 \subset P^2 \subset P^1,$$

which are still adapted to the biholomorphic equivalence problem for M . We use Cartan equivalence method, for which we refer to [10]. Eventually a Cartan connection is constructed on P^5 in section 4.

2. INITIAL G-STRUCTURE

Let M be a CR-manifold belonging to general class III_2 and \mathcal{L} be a local generator of the CR-bundle L of M . As M belongs to general class III_2 , the three vector fields $\mathcal{T}, \mathcal{S}, \mathcal{R}$, defined by:

$$\mathcal{T} := i[\mathcal{L}, \overline{\mathcal{L}}],$$

$$\mathcal{S} := [\mathcal{L}, \mathcal{T}],$$

$$\mathcal{R} := [\mathcal{L}, \mathcal{T}],$$

are such that the following biholomorphic invariant conditions hold:

$$\begin{aligned} 3 &= \text{rank}_{\mathbb{C}}(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}), & 4 &= \text{rank}_{\mathbb{C}}(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}), \\ 4 &= \text{rank}_{\mathbb{C}}(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \overline{\mathcal{T}}), & 5 &= \text{rank}_{\mathbb{C}}(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \mathcal{R}). \end{aligned}$$

As a result there exist two functions A and B such that:

$$\overline{\mathcal{T}} = A \cdot \mathcal{T} + B \cdot \mathcal{S}.$$

From the fact that $\overline{\overline{\mathcal{T}}} = \mathcal{S}$, the functions A and B satisfy the relations:

$$B\overline{B} = 1,$$

$$\overline{A} + \overline{B}A = 0.$$

There also exist three functions E, F, G , such that:

$$[\mathcal{L}, \mathcal{R}] = E \cdot \mathcal{T} + F \cdot \mathcal{S} + G \cdot \mathcal{R}.$$

The five functions A, B, E, F, G appear to be fundamental as all other Lie brackets between the vector fields $\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}$ and \mathcal{R} can be expressed in terms of these five functions and their $\{\mathcal{L}, \overline{\mathcal{L}}\}$ -derivatives.

In the case of an embedded CR-manifold $M \subset \mathbb{C}^4$, we can give an explicit formula for the fundamental vector field \mathcal{L} , and hence for the functions A, B, P, Q , in terms of a graphing function of M . We refer to [8] for details on this question. Let us just mention that the submanifold $M \subset \mathbb{C}^4$ is represented in local coordinates:

$$(z, w_1, w_2, w_3) = (x + iy, u_1 + iv_1, u_2 + iv_2, u_3 + iv_3),$$

as a graph:

$$v_1 = \phi_1(x, y, u_1, u_2, u_3),$$

$$v_2 = \phi_2(x, y, u_1, u_2, u_3),$$

$$v_3 = \phi_3(x, y, u_1, u_2, u_3).$$

There exists a unique local generator \mathcal{L} of $T^{1,0}M$ of the form:

$$\mathcal{L} = \frac{\partial}{\partial z} + A^1 \frac{\partial}{\partial u_1} + A^2 \frac{\partial}{\partial u_2} + A^3 \frac{\partial}{\partial u_3},$$

having conjugate:

$$\overline{\mathcal{L}} = \frac{\partial}{\partial \bar{z}} + \overline{A^1} \frac{\partial}{\partial u_1} + \overline{A^2} \frac{\partial}{\partial u_2} + \overline{A^3} \frac{\partial}{\partial u_3},$$

which is a generator of $T^{0,1}M$. The explicit expressions of the functions A^1 , A^2 and A^3 in terms of ϕ can be found in [8].

Returning to the general case of abstract CR-manifolds, let

$$\omega_0 := (\tau_0, \sigma_0, \rho_0, \zeta_0, \overline{\zeta}_0)$$

be the dual coframe of $(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{L}, \overline{\mathcal{L}})$. We have:

Lemma 1. [7]. *The structure equations enjoyed by ω_0 are of the form:*

$$\begin{aligned} d\tau_0 &= T \tau_0 \wedge \sigma_0 + Q \tau_0 \wedge \rho_0 + K \tau_0 \wedge \zeta_0 + G \tau_0 \wedge \overline{\zeta}_0 \\ &\quad + N \sigma_0 \wedge \rho_0 + \sigma_0 \wedge \zeta_0 + B \sigma_0 \wedge \overline{\zeta}_0, \\ d\sigma_0 &= S \tau_0 \wedge \sigma_0 + P \tau_0 \wedge \rho_0 + F \tau_0 \wedge \zeta_0 + J \tau_0 \wedge \overline{\zeta}_0 + M \sigma_0 \wedge \rho_0 \\ &\quad + (\mathcal{L}(B) + A) \sigma_0 \wedge \overline{\zeta}_0 + B \rho_0 \wedge \overline{\zeta}_0 + \rho_0 \wedge \zeta_0, \\ d\rho_0 &= R \tau_0 \wedge \sigma_0 + O \tau_0 \wedge \rho_0 + H \tau_0 \wedge \zeta_0 + E \tau_0 \wedge \overline{\zeta}_0 \\ &\quad + L \sigma_0 \wedge \rho_0 + \mathcal{L}(A) \sigma_0 \wedge \overline{\zeta}_0 + A \rho_0 \wedge \overline{\zeta}_0 + i \zeta_0 \wedge \overline{\zeta}_0, \\ d\zeta_0 &= 0, \\ d\overline{\zeta}_0 &= 0, \end{aligned}$$

where the twelve functions:

$$H, J, K, L, M, N, O, P, Q, R, S, T,$$

can be expressed in terms of the five fundamental functions:

$$A, B, E, F, G,$$

and their $\{\mathcal{L}, \overline{\mathcal{L}}\}$ -derivatives.

Let $h : M \longrightarrow M$ be a CR-automorphism of M . As we have

$$h_*(L) = L,$$

there exists a non-vanishing complex-valued function a on M such that:

$$h_*(\mathcal{L}) = a \mathcal{L}.$$

From the definition of $\mathcal{T}, \mathcal{S}, \mathcal{R}$ and the invariance

$$h_*([X, Y]) = [h_*(X), h_*(Y)]$$

for any vector fields X, Y on M , we easily get the existence of eight functions

$$b, c, d, e, f, g, h, k : M \longrightarrow \mathbb{C},$$

such that

$$h_* \begin{pmatrix} \mathcal{L} \\ \overline{\mathcal{L}} \\ \mathcal{T} \\ \mathcal{S} \\ \mathcal{R} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & a^2\bar{a} & 0 \\ k & h & g & f & a^3\bar{a} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{L} \\ \overline{\mathcal{L}} \\ \mathcal{T} \\ \mathcal{S} \\ \mathcal{R} \end{pmatrix}.$$

This is summarized in the following lemma:

Lemma 2. [6]. *Let $h : M \rightarrow M$ a CR-automorphism of M and let G_1 be the subgroup of $\mathrm{GL}_5(\mathbb{C})$:*

$$G_1 := \left\{ \begin{pmatrix} a^3\bar{a} & 0 & 0 & 0 & 0 \\ f & a^2\bar{a} & 0 & 0 & 0 \\ g & c & a\bar{a} & 0 & 0 \\ h & d & b & a & 0 \\ k & e & \bar{b} & 0 & \bar{a} \end{pmatrix}, a \in \mathbb{C} \setminus \{0\}, b, c, d, e, f, g, h, k \in \mathbb{C} \right\}.$$

Then the pullback ω of ω_0 by h , $\omega := h^*\omega_0$, satisfies:

$$\omega = g \cdot \omega_0,$$

where g is smooth (locally defined) function $M \xrightarrow{g} G_1$.

Let P^1 be the G_1 -structure on M defined by the coframes ω of the form

$$\omega := g \cdot \omega_0, \quad g \in G_1$$

The next section is devoted to construct four subgroups of G_1 :

$$G_5 \subset G_4 \subset G_3 \subset G_2 \subset G_1,$$

and four G_i -structures on M :

$$P^5 \subset P^4 \subset P^3 \subset P^2 \subset P^1,$$

which are adapted to the biholomorphic equivalence problem for M in the sense that a diffeomorphism h of M is a CR-automorphism if and only if h^* is a G_i -structure isomorphism of P^i .

3. REDUCTIONS OF P^1

The coframe ω_0 gives a natural (local) trivialisation $P^1 \xrightarrow{tr} M \times G_1$ from which we may consider any differential form on M (resp. G_1) as a differential form on P^1 through the pullback by the first (resp. the second) component of tr . With this identification, the structure equations of P^1 are naturally obtained by the formula:

$$(1) \quad d\omega = dg \cdot g^{-1} \wedge \omega + g \cdot d\omega_0.$$

The term $g \cdot d\omega_0$ contains the so-called torsion coefficients of P^1 . A 1-form $\tilde{\alpha}$ on P^1 is called a modified Maurer-Cartan form if its restriction to any fiber of P^1 is a Maurer-Cartan form of G_1 , or equivalently, if it is of the form:

$$\tilde{\alpha} := \alpha - x_\tau \tau - x_\sigma \sigma - x_\rho \rho - x_\zeta \zeta - x_{\bar{\zeta}} \bar{\zeta},$$

where $x_\sigma, x_\rho, x_\zeta, x_{\bar{\zeta}}$ are arbitrary complex-valued functions on M and where α is a Maurer-Cartan form of G_1 .

A basis for the Maurer-Cartan forms of G_1 is given by the following 1-forms:

$$\begin{aligned} \alpha^1 &:= \frac{da}{a}, \\ \alpha^2 &:= -\frac{bda}{a^2\bar{a}} + \frac{db}{a\bar{a}}, \\ \alpha^3 &:= -\frac{cda}{\bar{a}a^3} - \frac{cd\bar{a}}{\bar{a}^2a^2} + \frac{dc}{a^2\bar{a}}, \\ \alpha^4 &:= -\frac{(da\bar{a} - bc)da}{a^4\bar{a}^2} - \frac{cdb}{a^3\bar{a}^2} + \frac{dd}{a^2\bar{a}}, \\ \alpha^5 &:= -\frac{(ea\bar{a} - \bar{b}c)d\bar{a}}{a^3\bar{a}^3} - \frac{cd\bar{b}}{a^3\bar{a}^2} + \frac{de}{a^2\bar{a}}, \\ \alpha^6 &:= -2\frac{fda}{\bar{a}a^4} - \frac{fd\bar{a}}{a^3\bar{a}^2} + \frac{df}{\bar{a}a^3}, \\ \alpha^7 &:= -\frac{(ga^2\bar{a} - cf)da}{\bar{a}^2a^6} - \frac{(ga^2\bar{a} - cf)d\bar{a}}{\bar{a}^3a^5} - \frac{fdc}{a^5\bar{a}^2} + \frac{dg}{\bar{a}a^3}, \\ \alpha^8 &:= -\frac{(ha^3\bar{a}^2 - dfa\bar{a} - bga^2\bar{a} + bcf)da}{a^7\bar{a}^3} - \frac{(ga^2\bar{a} - cf)db}{a^6\bar{a}^3} - \frac{fdd}{a^5\bar{a}^2} + \frac{dh}{\bar{a}a^3}, \\ \alpha^9 &:= -\frac{(ka^3\bar{a}^2 - efa\bar{a} - \bar{b}ga^2\bar{a} + \bar{b}cf)d\bar{a}}{a^6\bar{a}^4} - \frac{(ga^2\bar{a} - cf)d\bar{b}}{a^6\bar{a}^3} - \frac{fde}{a^5\bar{a}^2} + \frac{dk}{\bar{a}a^3}, \end{aligned}$$

together with their conjugates.

We derive the structure equations of P^1 from the relations (1). The expression of $d\tau$ is:

$$\begin{aligned} d\tau &= 3\alpha^1 \wedge \tau + \bar{\alpha}^1 \wedge \tau \\ &\quad + T_{\tau\sigma}^\tau \tau \wedge \sigma + T_{\tau\rho}^\tau \tau \wedge \rho + T_{\tau\zeta}^\tau \tau \wedge \zeta \\ &\quad + T_{\tau\bar{\zeta}}^\tau \tau \wedge \bar{\zeta} + T_{\sigma\rho}^\tau \sigma \wedge \rho + \sigma \wedge \zeta - \frac{a}{\bar{a}} B \sigma \wedge \bar{\zeta} \end{aligned}$$

The coefficient

$$\frac{a}{\bar{a}} B,$$

which can not be absorbed for any choice of the modified Maurer-Cartan form $\tilde{\alpha}^1$, is referred to as an essential torsion coefficient. From standard

results on Cartan theory (see [10, 15]), a diffeomorphism of M is an isomorphism of the G_1 -structure P^1 if and only if it is an isomorphism of the reduced bundle $P^2 \subset P^1$ consisting of those coframes ω on M such that

$$\frac{a}{\bar{a}} B = 1.$$

This is equivalent to the normalization:

$$\bar{a} = aB.$$

A coframe $\omega \in P^2$ is related to the coframe ω_0 by the relations:

$$\begin{aligned} \tau &= a^4 B \tau_0, & \sigma &= f \tau_0 + a^3 B \sigma_0, \\ \rho &= g \tau_0 + c \sigma_0 + a^2 B \rho_0, & \zeta &= h \tau_0 + d \sigma_0 + b \rho_0 + a \zeta_0, \\ \bar{\zeta} &= k \tau_0 + e \sigma_0 + \bar{b} \rho_0 + a B \bar{\zeta}_0, \end{aligned}$$

which are equivalent to:

$$\begin{aligned} \tau &= a'^4 \tau_1, & \sigma &= f' \tau_1 + a'^3 \sigma_1, \\ \rho &= g' \tau_1 + c' \sigma_1 + a'^2 \rho_1, & \zeta &= h' \tau_1 + d' \sigma_1 + b \rho_1 + a' \zeta_1, \\ \bar{\zeta} &= k' \tau_1 + e' \sigma_1 + \bar{b} \rho_1 + a' \bar{\zeta}_1, \end{aligned}$$

where:

$$\tau_1 := \frac{\tau_0}{B}, \quad \sigma_1 := \frac{\sigma_0}{B^{\frac{1}{2}}}, \quad \rho_1 = \rho_0, \quad \zeta_1 := \frac{\zeta_0}{B^{\frac{1}{2}}},$$

and

$$x' := \begin{cases} x \cdot B^{\frac{1}{2}}, & \text{for } x = a, c, d, e, \\ x \cdot B, & \text{for } x = f, g, h, k. \end{cases}$$

We notice that a' is a real parameter, and that τ_1 is a real 1-form. Let ω_1 be the coframe $\omega_1 := (\tau_1, \sigma_1, \rho_1, \zeta_1, \bar{\zeta}_1)$, and G_2 be the subgroup of G_1 :

$$G_2 := \left\{ \begin{pmatrix} a^4 & 0 & 0 & 0 & 0 \\ f & a^3 & 0 & 0 & 0 \\ g & c & a^2 & 0 & 0 \\ h & d & b & a & 0 \\ k & e & \bar{b} & 0 & a \end{pmatrix}, a \in \mathbb{R} \setminus \{0\}, b, c, d, e, f, g, h, k \in \mathbb{C} \right\}.$$

A coframe ω on M belongs to P^2 if and only if there is a local function $g : M \xrightarrow{g} G_2$ such that $\omega = g \cdot \omega_1$, namely P^2 is a G_2 structure on M .

The Maurer-Cartan forms of G_2 are given by:

$$\begin{aligned}
\beta^1 &:= \frac{da}{a}, \\
\beta^2 &:= -\frac{bda}{a^3} + \frac{db}{a^2}, \\
\beta^3 &:= -2\frac{cda}{a^4} + \frac{dc}{a^3}, \\
\beta^4 &= -\frac{(da^2 - bc)da}{a^6} - \frac{cdb}{a^5} + \frac{dd}{a^3}, \\
\beta^5 &= -\frac{(ea^2 - \bar{b}c)da}{a^6} - \frac{cd\bar{b}}{a^5} + \frac{de}{a^3}, \\
\beta^6 &= -3\frac{fda}{a^5} + \frac{df}{a^4}, \\
\beta^7 &= -2\frac{(ga^3 - cf)da}{a^8} - \frac{fdc}{a^7} + \frac{dg}{a^4}, \\
\beta^8 &= -\frac{(ha^5 - dfa^2 - bga^3 + bcf)da}{a^{10}} - \frac{(ga^3 - cf)db}{a^9} - \frac{fdd}{a^7} + \frac{dh}{a^4}, \\
\beta^9 &= -\frac{(ka^5 - efa^2 - \bar{b}ga^3 + \bar{b}cf)da}{a^{10}} - \frac{(ga^3 - cf)d\bar{b}}{a^9} - \frac{fde}{a^5\bar{a}^2} + \frac{dk}{a^4},
\end{aligned}$$

together with $\bar{\beta}^i$, $i = 2 \dots 9$.

Using formula (1), we get the structure equations of P^2 :

$$\begin{aligned}
d\tau &= 4\beta^1 \wedge \tau \\
&\quad + U_{\tau\sigma}^\tau \tau \wedge \sigma + U_{\tau\rho}^\tau \tau \wedge \rho + U_{\tau\zeta}^\tau \tau \wedge \zeta + U_{\tau\bar{\zeta}}^\tau \tau \wedge \bar{\zeta} \\
&\quad + U_{\sigma\rho}^\tau \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \bar{\zeta},
\end{aligned}$$

$$\begin{aligned}
d\sigma &= 3\beta^1 \wedge \sigma + \beta^6 \wedge \tau \\
&\quad + U_{\tau\sigma}^\sigma \tau \wedge \sigma + U_{\tau\rho}^\sigma \tau \wedge \rho + U_{\tau\zeta}^\sigma \tau \wedge \zeta \\
&\quad + U_{\tau\bar{\zeta}}^\sigma \tau \wedge \bar{\zeta} + U_{\sigma\rho}^\sigma \sigma \wedge \rho + U_{\sigma\zeta}^\sigma \sigma \wedge \zeta \\
&\quad + U_{\sigma\bar{\zeta}}^\sigma \sigma \wedge \bar{\zeta} + \rho \wedge \zeta + \rho \wedge \bar{\zeta}
\end{aligned}$$

$$\begin{aligned}
d\rho &= 2\beta^1 \wedge \rho + \beta^3 \wedge \sigma + \beta^7 \wedge \tau \\
&\quad + U_{\tau\sigma}^\rho \tau \wedge \sigma + U_{\tau\rho}^\rho \tau \wedge \rho + U_{\tau\zeta}^\rho \tau \wedge \zeta + U_{\tau\bar{\zeta}}^\rho \tau \wedge \bar{\zeta} + U_{\sigma\rho}^\rho \sigma \wedge \rho \\
&\quad + U_{\sigma\zeta}^\rho \sigma \wedge \zeta + U_{\sigma\bar{\zeta}}^\rho \sigma \wedge \bar{\zeta} + U_{\rho\zeta}^\rho \rho \wedge \zeta + U_{\rho\bar{\zeta}}^\rho \rho \wedge \bar{\zeta} + i\zeta \wedge \bar{\zeta},
\end{aligned}$$

$$\begin{aligned}
d\zeta = & \beta^1 \wedge \zeta + \beta^2 \wedge \rho + \beta^4 \wedge \sigma + \beta^8 \wedge \tau \\
& + U_{\tau\sigma}^\zeta \tau \wedge \sigma + U_{\tau\rho}^\zeta \tau \wedge \rho + U_{\tau\zeta}^\zeta \tau \wedge \zeta + U_{\tau\bar{\zeta}}^\zeta \tau \wedge \bar{\zeta} \\
& + U_{\sigma\rho}^\zeta \sigma \wedge \rho + U_{\sigma\zeta}^\zeta \sigma \wedge \zeta + U_{\sigma\bar{\zeta}}^\zeta \sigma \wedge \bar{\zeta} + U_{\rho\zeta}^\zeta \rho \wedge \zeta \\
& + U_{\rho\bar{\zeta}}^\zeta \rho \wedge \bar{\zeta} + U_{\zeta\bar{\zeta}}^\zeta \zeta \wedge \bar{\zeta}.
\end{aligned}$$

Introducing the modified Maurer-Cartan forms:

$$\tilde{\beta}^i = \beta^i - y_\tau^i \tau - y_\sigma^i \sigma - y_\rho^i \rho - y_\zeta^i \zeta - y_{\bar{\zeta}}^i \bar{\zeta},$$

the structure equations rewrite:

$$\begin{aligned}
d\tau = & 4\tilde{\beta}^1 \wedge \tau \\
& + (U_{\tau\sigma}^\tau - 4y_\sigma^1) \tau \wedge \sigma + (U_{\tau\rho}^\tau - 4y_\rho^1) \tau \wedge \rho \\
& + (U_{\tau\zeta}^\tau - 4y_\zeta^1) \tau \wedge \zeta + (U_{\tau\bar{\zeta}}^\tau - 4y_{\bar{\zeta}}^1) \tau \wedge \bar{\zeta} \\
& + U_{\sigma\rho}^\tau \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \bar{\zeta},
\end{aligned}$$

$$\begin{aligned}
d\sigma = & 3\tilde{\beta}^1 \wedge \sigma + \tilde{\beta}^6 \wedge \tau \\
& + (U_{\tau\sigma}^\sigma + 3y_\tau^1 - y_\sigma^6) \tau \wedge \sigma + (U_{\tau\rho}^\sigma - y_\rho^6) \tau \wedge \rho \\
& + (U_{\tau\zeta}^\sigma - y_\zeta^6) \tau \wedge \zeta + (U_{\tau\bar{\zeta}}^\sigma - y_{\bar{\zeta}}^6) \tau \wedge \bar{\zeta} \\
& + (U_{\sigma\rho}^\sigma - 3y_\rho^1) \sigma \wedge \rho + (U_{\sigma\zeta}^\sigma - 3y_\zeta^1) \sigma \wedge \zeta \\
& + (U_{\sigma\bar{\zeta}}^\sigma - 3y_{\bar{\zeta}}^1) \sigma \wedge \bar{\zeta} + \rho \wedge \zeta + \rho \wedge \bar{\zeta}
\end{aligned}$$

$$\begin{aligned}
d\rho = & 2\tilde{\beta}^1 \wedge \rho + \tilde{\beta}^3 \wedge \sigma + \tilde{\beta}^7 \wedge \tau \\
& + (U_{\tau\sigma}^\rho + y_\tau^3 - y_\sigma^7) \tau \wedge \sigma + (U_{\tau\rho}^\rho + 2y_\tau^1 - y_\rho^7) \tau \wedge \rho \\
& + (U_{\tau\zeta}^\rho - y_\zeta^7) \tau \wedge \zeta + (U_{\tau\bar{\zeta}}^\rho - y_{\bar{\zeta}}^7) \tau \wedge \bar{\zeta} \\
& + (U_{\sigma\rho}^\rho + 2y_\sigma^1 - y_\rho^3) \sigma \wedge \rho + (U_{\sigma\zeta}^\rho - y_\zeta^3) \sigma \wedge \zeta \\
& + (U_{\sigma\bar{\zeta}}^\rho - y_{\bar{\zeta}}^3) \sigma \wedge \bar{\zeta} + (U_{\rho\zeta}^\rho - 2y_\zeta^1) \rho \wedge \zeta \\
& + (U_{\rho\bar{\zeta}}^\rho - 2y_{\bar{\zeta}}^1) \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta},
\end{aligned}$$

$$\begin{aligned}
d\zeta = & \widetilde{\beta}^1 \wedge \zeta + \widetilde{\beta}^2 \wedge \rho + \widetilde{\beta}^4 \wedge \sigma + \widetilde{\beta}^8 \wedge \tau \\
& + (U_{\tau\sigma}^\zeta + y_\tau^4 - y_\sigma^8) \tau \wedge \sigma + (U_{\tau\rho}^\zeta + y_\tau^2 - y_\rho^8) \tau \wedge \rho \\
& + (U_{\tau\zeta}^\zeta + y_\tau^1 - y_\zeta^8) \tau \wedge \zeta + (U_{\tau\bar{\zeta}}^\zeta - y_{\bar{\zeta}}^8) \tau \wedge \bar{\zeta} \\
& + (U_{\sigma\rho}^\zeta + y_\sigma^2 - y_\rho^4) \sigma \wedge \rho + (U_{\sigma\zeta}^\zeta + y_\sigma^1 - y_\zeta^4) \sigma \wedge \zeta \\
& + (U_{\sigma\bar{\zeta}}^\zeta - y_{\bar{\zeta}}^4) \sigma \wedge \bar{\zeta} + (U_{\rho\zeta}^\zeta + y_\rho^1 - y_\zeta^2) \rho \wedge \zeta \\
& + (U_{\rho\bar{\zeta}}^\zeta - y_{\bar{\zeta}}^2) \rho \wedge \bar{\zeta} + (U_{\zeta\bar{\zeta}}^\zeta - y_{\bar{\zeta}}^1) \zeta \wedge \bar{\zeta}.
\end{aligned}$$

We get the following absorbtion equations:

$$\begin{array}{lll}
4 y_\sigma^1 = U_{\tau\sigma}^\tau, & 4 y_\rho^1 = U_{\tau\rho}^\tau, & 4 y_\zeta^1 = U_{\tau\zeta}^\tau, \\
4 y_{\bar{\zeta}}^1 = U_{\tau\bar{\zeta}}^\tau, & -3 y_\tau^1 + y_\sigma^6 = U_{\tau\sigma}^\sigma, & y_\rho^6 = U_{\tau\rho}^\sigma, \\
y_\zeta^6 = U_{\tau\zeta}^\sigma, & y_{\bar{\zeta}}^6 = U_{\tau\bar{\zeta}}^\sigma, & 3 y_\rho^1 = U_{\sigma\rho}^\sigma, \\
3 y_\zeta^1 = U_{\sigma\zeta}^\sigma, & 3 y_{\bar{\zeta}}^1 = U_{\sigma\bar{\zeta}}^\sigma, & -y_\tau^3 + y_\sigma^7 = U_{\tau\sigma}^\rho, \\
-2 y_\tau^1 + y_\rho^7 = U_{\tau\rho}^\rho, & y_\zeta^7 = U_{\tau\zeta}^\rho, & y_{\bar{\zeta}}^7 = U_{\tau\bar{\zeta}}^\rho, \\
-2 y_\sigma^1 + y_\rho^3 = U_{\sigma\rho}^\rho, & y_{\bar{\zeta}}^3 = U_{\sigma\bar{\zeta}}^\rho, & y_{\bar{\zeta}}^3 = U_{\sigma\bar{\zeta}}^\rho, \\
2 y_\zeta^1 = U_{\rho\zeta}^\rho, & 2 y_{\bar{\zeta}}^1 = U_{\rho\bar{\zeta}}^\rho, & -y_\tau^4 + y_\sigma^8 = U_{\tau\sigma}^\zeta, \\
-y_\tau^2 + y_\rho^8 = U_{\tau\rho}^\zeta, & -y_\tau^1 + y_\zeta^8 = U_{\tau\zeta}^\zeta, & y_{\bar{\zeta}}^8 = U_{\tau\bar{\zeta}}^\zeta, \\
-y_\sigma^2 + y_\rho^4 = U_{\sigma\rho}^\zeta, & -y_\sigma^1 + y_\zeta^4 = U_{\sigma\zeta}^\zeta, & y_{\bar{\zeta}}^4 = U_{\sigma\bar{\zeta}}^\zeta, \\
-y_\rho^1 + y_\zeta^2 = U_{\rho\zeta}^\zeta, & y_{\bar{\zeta}}^2 = U_{\rho\bar{\zeta}}^\zeta, & y_{\bar{\zeta}}^1 = U_{\zeta\bar{\zeta}}^\zeta.
\end{array}$$

Eliminating $y_{\zeta\tau}^1$ and $y_{\bar{\zeta}}^1$ among the previous equations leads to the normalizations:

$$\begin{aligned}
b &= a B_0, \\
c &= a^2 C_0, \\
f &= a^3 F_0,
\end{aligned}$$

where the functions B_0 , C_0 and F_0 are defined by:

$$\begin{aligned}
\mathbf{B}_0 &:= \frac{3i}{10} \frac{\overline{\mathcal{L}}(B)}{B^{\frac{3}{2}}} - \frac{i}{5} \frac{A}{B^{\frac{1}{2}}} - \frac{i}{10} \frac{K}{B^{\frac{1}{2}}} - \frac{i}{10} \frac{\mathcal{L}(B)}{B^{\frac{1}{2}}}, \\
\mathbf{C}_0 &:= \frac{11}{20} \frac{\mathcal{L}(B)}{B^{\frac{1}{2}}} + \frac{3}{20} B^{\frac{1}{2}} G + \frac{1}{20} \frac{\overline{\mathcal{L}}(B)}{B^{\frac{3}{2}}} - \frac{1}{5} \frac{A}{B^{\frac{1}{2}}} + \frac{3}{20} \frac{K}{B^{\frac{1}{2}}}, \\
\mathbf{F}_0 &:= \frac{1}{10} \frac{\mathcal{L}(B)}{B} + \frac{3}{10} B^{\frac{1}{2}} G + \frac{1}{10} \frac{\overline{\mathcal{L}}(B)}{B^{\frac{3}{2}}} - \frac{2}{5} \frac{A}{B^{\frac{1}{2}}} + \frac{3}{10} \frac{K}{B^{\frac{1}{2}}}.
\end{aligned}$$

The absorbed structure equations take the form:

$$\begin{aligned}
d\tau &= 4\tilde{\beta}^1 \wedge \tau + \frac{\mathfrak{I}_1}{a} \tau \wedge \zeta - \frac{\mathfrak{I}_1}{a} \tau \wedge \bar{\zeta} + 3 \frac{\mathfrak{I}_1}{a} \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \bar{\zeta}, \\
d\sigma &= 3 \tilde{\beta}^1 \wedge \sigma + \tilde{\beta}^6 \wedge \tau - \frac{\mathfrak{I}_1}{2a} \sigma \wedge \zeta + \frac{\mathfrak{I}_1}{2a} \sigma \wedge \bar{\zeta} + \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \\
d\rho &= 2\tilde{\beta}^1 \wedge \rho + \tilde{\beta}^3 \wedge \sigma + \tilde{\beta}^7 \wedge \tau - \frac{\mathfrak{I}_1}{2a} \rho \wedge \zeta + \frac{\mathfrak{I}_1}{2a} \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta}, \\
d\zeta &= \tilde{\beta}^1 \wedge \zeta + \tilde{\beta}^2 \wedge \rho + \tilde{\beta}^4 \wedge \sigma + \tilde{\beta}^8 \wedge \tau,
\end{aligned}$$

where the function \mathfrak{I}_1 is a biholomorphic invariant of M and is given by:

$$\mathfrak{I}_1 := \frac{1}{2} \frac{\mathcal{L}(B)}{B} + \frac{3}{10} B^{\frac{1}{2}} G - \frac{1}{10} \frac{\overline{\mathcal{L}}(B)}{B^{\frac{3}{2}}} + \frac{2}{5} \frac{A}{B^{\frac{1}{2}}} - \frac{3}{10} \frac{K}{B^{\frac{1}{2}}}.$$

We introduce the coframe $\omega_2 := (\tau_2, \sigma_2, \rho_2, \zeta_2, \bar{\zeta}_2)$ on M , defined by:

$$\begin{cases} \tau_2 := \tau_1 \\ \sigma_2 := \mathbf{F}_0 \tau_1 + \sigma_1, \\ \rho_2 := \rho_1 + \mathbf{C}_0 \sigma_1, \\ \zeta_2 := \zeta_1 + \mathbf{B}_0 \rho_1, \end{cases}$$

and the subgroup $G_3 \subset G_2$:

$$G_3 := \left\{ \begin{pmatrix} a^4 & 0 & 0 & 0 & 0 \\ 0 & a^3 & 0 & 0 & 0 \\ g & 0 & a^2 & 0 & 0 \\ h & d & 0 & a & 0 \\ k & e & 0 & 0 & a \end{pmatrix}, a \in \mathbb{R} \setminus \{0\}, d, e, g, h, k, \in \mathbb{C} \right\}.$$

We notice that σ_2 is a real one-form. The normalizations:

$$b := a \mathbf{B}_0, \quad c := a^2 \mathbf{C}_0, \quad f := a^3 \mathbf{F}_0,$$

amount to consider the subbundle $P^3 \subset P^2$ consisting of those coframes ω of the form

$$\omega := g \cdot \omega_2, \quad \text{where } g \text{ is a function } g : M \xrightarrow{g} G_3.$$

A basis of the Maurer Cartan forms of G_3 is given by:

$$\begin{aligned}\gamma^1 &:= \frac{da}{a}, \\ \gamma^2 &:= -\frac{dda}{a^4} + \frac{dd}{a^3}, \\ \gamma^3 &:= -\frac{eda}{a^4} + \frac{de}{a^3}, \\ \gamma^4 &:= -2\frac{gda}{a^5} + \frac{dg}{a^4}, \\ \gamma^5 &:= -\frac{hda}{a^5} + \frac{dh}{a^4}, \\ \gamma^6 &:= -\frac{kda}{a^5} + \frac{dk}{a^4}.\end{aligned}$$

We get the following absorbed structure equations for P^3 :

$$d\tau = 4\tilde{\gamma}^1 \wedge \tau + \frac{\mathfrak{I}_1}{a} \tau \wedge \zeta - \frac{\mathfrak{I}_1}{a} \tau \wedge \bar{\zeta} + 3\frac{\mathfrak{I}_1}{a} \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \bar{\zeta},$$

$$\begin{aligned}d\sigma &= 3\tilde{\gamma}^1 \wedge \sigma \\ &\quad + V_{\tau\rho}^\sigma \tau \wedge \rho + V_{\tau\zeta}^\sigma \tau \wedge \zeta + V_{\tau\bar{\zeta}}^\sigma \tau \wedge \bar{\zeta} + V_{\sigma\rho}^\sigma \sigma \wedge \rho \\ &\quad - \frac{\mathfrak{I}_1}{2a} \sigma \wedge \zeta + \frac{\mathfrak{I}_1}{2a} \sigma \wedge \bar{\zeta} + \rho \wedge \zeta + \rho \wedge \bar{\zeta},\end{aligned}$$

$$\begin{aligned}d\rho &= 2\tilde{\gamma}^1 \wedge \rho + \tilde{\gamma}^4 \wedge \tau \\ &\quad + V_{\sigma\rho}^\rho \sigma \wedge \rho + V_{\sigma\zeta}^\rho \sigma \wedge \zeta + V_{\sigma\bar{\zeta}}^\rho \sigma \wedge \bar{\zeta} \\ &\quad + \frac{\mathfrak{I}_1}{2a} + \rho \wedge \zeta + \frac{\mathfrak{I}_1}{2a} \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta},\end{aligned}$$

$$d\zeta = \tilde{\gamma}^1 \wedge \zeta + \tilde{\gamma}^2 \wedge \sigma + \tilde{\gamma}^5 \wedge \tau + V_{\rho\zeta}^\zeta \rho \wedge \zeta + V_{\rho\bar{\zeta}}^\zeta \rho \wedge \bar{\zeta},$$

From the essential torsion coefficients $V_{\tau\zeta}^\sigma$, $V_{\tau\bar{\zeta}}^\sigma$ and $V_{\rho\zeta}^\zeta$, we obtain the normalizations:

$$d := a D_0, \quad g := a^2 G_0,$$

where

$$D_0 := i B_0^2 - \frac{AB_0}{B^{\frac{1}{2}}} + \frac{\overline{\mathcal{L}}(B_0)}{B^{\frac{1}{2}}} + \frac{1}{2} \frac{\overline{\mathcal{L}}(B)B_0}{B^{\frac{3}{2}}},$$

and

$$\begin{aligned} \mathbf{G}_0 := & -\frac{1}{4} \frac{\mathcal{L}(B)}{B^{\frac{1}{2}}} \mathbf{F}_0 - \mathbf{F}_0^2 + \frac{1}{2} B^{\frac{1}{2}} G \mathbf{F}_0 - \frac{1}{2} B^{\frac{1}{2}} \mathcal{L}(\mathbf{F}_0) + \mathbf{C}_0 \mathbf{F}_0 \\ & + \frac{1}{2} F B + \frac{1}{4} \frac{\overline{\mathcal{L}}(B)}{B^{\frac{3}{2}}} \mathbf{F}_0 + \frac{1}{2} \frac{K}{B^{\frac{1}{2}}} \mathbf{F}_0 - \frac{1}{2} \frac{\overline{\mathcal{L}}(\mathbf{F}_0)}{B^{\frac{1}{2}}} + \frac{J}{2} - \frac{1}{2} \frac{A}{B^{\frac{1}{2}}} \mathbf{F}_0. \end{aligned}$$

We introduce the coframe $\omega_3 := (\tau_3, \sigma_3, \rho_3, \zeta_3, \bar{\zeta}_3)$ on M , defined by:

$$\begin{cases} \tau_3 := \tau_2 \\ \sigma_3 := \sigma_2 \\ \rho_3 := \rho_2 + \mathbf{C}_0 \tau_2, \\ \zeta_3 := \zeta_2 + \mathbf{D}_0 \sigma_2, \end{cases}$$

and the subgroup $G_4 \subset G_3$:

$$G_4 := \left\{ \begin{pmatrix} \mathbf{a}^4 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}^3 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a}^2 & 0 & 0 \\ \mathbf{h} & 0 & 0 & \mathbf{a} & 0 \\ \bar{\mathbf{h}} & 0 & 0 & 0 & \mathbf{a} \end{pmatrix}, \mathbf{a} \in \mathbb{R} \setminus \{0\}, \mathbf{h} \in \mathbb{C} \right\}$$

The normalizations:

$$\mathbf{d} := \mathbf{a} \mathbf{D}_0, \quad \mathbf{g} := \mathbf{a}^2 \mathbf{G}_0,$$

amount to consider the subbundle $P^4 \subset P^3$ consisting of those coframes ω of the form

$$\omega := g \cdot \omega_3, \quad \text{where } g \text{ is a function } g : M \xrightarrow{g} G_4.$$

A basis of the Maurer-Cartan forms is given by:

$$\begin{aligned} \delta^1 &:= \frac{d\mathbf{a}}{\mathbf{a}}, \\ \delta^2 &:= -\frac{\mathbf{h} d\mathbf{a}}{\mathbf{a}^5} + \frac{d\mathbf{h}}{\mathbf{a}^4}, \end{aligned}$$

together with $\bar{\delta}^2$.

As for the previous step, we determine the structure equations of P^4 using formula (1). We just write here the expression of $d\zeta$, as it provides a normalization of \mathbf{h} :

$$d\zeta = \tilde{\delta}^1 \wedge \zeta + \tilde{\delta}^2 \wedge \tau + W_{\sigma\rho}^\zeta \sigma \wedge \rho + W_{\sigma\zeta}^\zeta \sigma \wedge \zeta + W_{\sigma\bar{\zeta}}^\zeta \sigma \wedge \bar{\zeta},$$

for some modified Maurer-Cartan forms $\tilde{\delta}^1, \tilde{\delta}^2$.

The essential torsion coefficient $W_{\sigma\zeta}^\zeta$ can be normalized to 0, which is equivalent to the normalization:

$$\mathbf{h} := \mathbf{a} \mathbf{H}_0,$$

where

$$\begin{aligned} \mathbf{H}_0 := & -\mathbf{D}_0 \mathbf{F}_0 + \mathbf{C}_0 \mathbf{D}_0 - \frac{\mathcal{L}(B)}{B^{\frac{1}{2}}} \mathbf{D}_0 - \frac{A}{B^{\frac{1}{2}}} \mathbf{D}_0 + \overline{\mathcal{L}}(\mathbf{D}_0) B^{\frac{1}{2}} + i \mathbf{B}_0 \mathbf{D}_0 \\ & - i \mathbf{B}_0^2 \mathbf{C}_0 + \frac{A}{B^{\frac{1}{2}}} \mathbf{B}_0 \mathbf{C}_0 - \mathcal{L}(A) \mathbf{B}_0 - \frac{\overline{\mathcal{L}}(\mathbf{B}_0)}{B^{\frac{1}{2}}} \mathbf{C}_0 - \frac{1}{2} \frac{\overline{\mathcal{L}}(B)}{B^{\frac{3}{2}}} \mathbf{B}_0 \mathbf{C}_0. \end{aligned}$$

Let G_5 be the 1-dimensional Lie subgroup of G_4 whose elements g are of the form:

$$g := \begin{pmatrix} \mathbf{a}^4 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}^3 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a}^2 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a} \end{pmatrix}, \quad \mathbf{a} \in \mathbb{R} \setminus \{0\},$$

and let $\omega_4 := (\tau_4, \sigma_4, \rho_4, \zeta_4, \bar{\zeta}_4)$ be the coframe defined on M by:

$$\sigma_4 := \sigma_3, \quad \rho_4 := \rho_3, \quad \zeta_4 := \zeta_3 + \mathbf{H}_0 \tau_3.$$

The normalization of \mathbf{h} is equivalent to the reduction of P^4 to a subbundle P^5 consisting of those coframes ω on M such that:

$$\omega := g \cdot \omega_3, \quad \text{where } g \text{ is a function } g : M \xrightarrow{g} G_4.$$

The Maurer-Cartan forms of G_5 are spanned by:

$$\alpha := \frac{da}{a}.$$

Proceeding as in the previous steps, we determine the structure equations of P^4 which take the absorbed form:

$$d\tau = 4\Lambda \wedge \tau + \frac{\mathfrak{I}_1}{a} \tau \wedge \zeta - \frac{\mathfrak{I}_1}{a} \tau \wedge \bar{\zeta} + 3 \frac{\mathfrak{I}_1}{a} \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \bar{\zeta},$$

$$d\sigma = 3\Lambda \wedge \sigma$$

$$\begin{aligned} & + \frac{\mathfrak{I}_2}{a^3} \tau \wedge \rho + \frac{\mathfrak{I}_3}{a^2} \tau \wedge \zeta + \frac{\bar{\mathfrak{I}}_3}{a^2} \tau \wedge \bar{\zeta} + \frac{\mathfrak{I}_4}{a^2} \sigma \wedge \rho \\ & - \frac{\mathfrak{I}_1}{2a} \sigma \wedge \zeta + \frac{\mathfrak{I}_1}{2a} \sigma \wedge \bar{\zeta} + \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \end{aligned}$$

$$d\rho = 2\Lambda \wedge \rho$$

$$\begin{aligned} & + \frac{\mathfrak{I}_5}{a^5} \tau \wedge \sigma + \frac{\mathfrak{I}_6}{a^4} \tau \wedge \rho + \frac{\mathfrak{I}_7}{a^3} \tau \wedge \zeta + \frac{\bar{\mathfrak{I}}_7}{a^3} \tau \wedge \bar{\zeta} + \frac{\mathfrak{I}_8}{a^3} \sigma \wedge \rho \\ & + \frac{\mathfrak{I}_9}{a^2} \sigma \wedge \zeta + \frac{\bar{\mathfrak{I}}_9}{a^2} \sigma \wedge \bar{\zeta} - \frac{\mathfrak{I}_1}{2a} \rho \wedge \zeta + \frac{\mathfrak{I}_1}{2a} \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta}, \end{aligned}$$

$$\begin{aligned}
d\zeta = \Lambda \wedge \zeta \\
+ \frac{\mathfrak{I}_{10}}{a^6} \tau \wedge \sigma + \frac{\mathfrak{I}_{11}}{a^5} \tau \wedge \rho + \frac{\mathfrak{I}_{12}}{a^4} \tau \wedge \zeta + \frac{\mathfrak{I}_{13}}{a^4} \tau \wedge \bar{\zeta} \\
+ \frac{\mathfrak{I}_{14}}{a^4} \sigma \wedge \rho + \frac{\mathfrak{I}_{15}}{a^3} \sigma \wedge \zeta,
\end{aligned}$$

where Λ is a modified-Maurer Cartan form:

$$\Lambda := \frac{da}{a} - X_\tau \tau - X_\sigma \sigma - X_\rho \rho - X_\zeta \zeta - X_{\bar{\zeta}} \bar{\zeta},$$

and where

$$\mathfrak{I}_i, \quad i = 1 \dots 15,$$

are biholomorphic invariants of M .

The exterior derivative of Λ can be determined by taking the exterior derivative of the four previous equations which leads to the so-called Bianchi-Cartan's identities. We obtain the fact that $d\Lambda$ does not contain any 2-form involving the 1-form Λ , namely:

$$(3) \quad d\Lambda = \sum_{\nu\mu} X_{\nu\mu} \nu \wedge \mu, \quad \nu, \mu = \tau, \sigma, \rho, \zeta, \bar{\zeta}.$$

4. CARTAN CONNECTION

We recall that the model for CR-manifolds belonging to general class III₂ is the CR-manifold defined by the equations:

$$\begin{aligned}
N : \quad w_1 &= \overline{w_1} + 2i z \bar{z}, \\
w_2 &= \overline{w_2} + 2i z \bar{z} (z + \bar{z}), \\
w_3 &= \overline{w_3} + 2i z \bar{z} \left(z^2 + \frac{3}{2} z \bar{z} + \bar{z}^2 \right).
\end{aligned}$$

Its Lie algebra of infinitesimal CR-automorphisms is given by the following theorem:

Theorem 2. [12]. *The model of the class III₂:*

$$\begin{aligned}
N : \quad w_1 &= \overline{w_1} + 2i z \bar{z}, \\
w_2 &= \overline{w_2} + 2i z \bar{z} (z + \bar{z}), \\
w_3 &= \overline{w_3} + 2i z \bar{z} \left(z^2 + \frac{3}{2} z \bar{z} + \bar{z}^2 \right),
\end{aligned}$$

has a 6-dimensional Lie algebra of CR-automorphisms $\text{aut}_{\text{CR}}(N)$. A basis for the Maurer-Cartan forms of $\text{aut}_{\text{CR}}(N)$ is provided by the 6 differential

1-forms $\tau, \sigma, \rho, \zeta, \bar{\zeta}, \alpha$, which satisfy the Maurer-Cartan equations:

$$\begin{aligned} d\tau &= 4\alpha \wedge \tau + \sigma \wedge \zeta + \sigma \wedge \bar{\zeta}, \\ d\sigma &= 3\alpha \wedge \sigma + \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \\ d\rho &= 2\alpha \wedge \rho + i\zeta \wedge \bar{\zeta}, \\ d\zeta &= \alpha \wedge \zeta, \\ d\bar{\zeta} &= \alpha \wedge \bar{\zeta}, \\ d\alpha &= 0. \end{aligned}$$

Let us write \mathfrak{g} instead of $\text{aut}_{\text{CR}}(\mathbb{N})$ for the Lie algebra of infinitesimal automorphisms of \mathbb{N} and let $(e_\alpha, e_\tau, e_\sigma, e_\rho, e_\zeta, e_{\bar{\zeta}})$ be the dual basis of the basis of Maurer-Cartan 1-forms: $(\alpha, \tau, \sigma, \rho, \zeta, \bar{\zeta})$. From the above structure equations, the Lie brackets structure of \mathfrak{g} is given by:

$$\begin{aligned} [e_\alpha, e_\tau] &= -4e_\tau, & [e_\sigma, e_\zeta] &= -e_\tau, & [e_\sigma, e_{\bar{\zeta}}] &= -e_\tau, \\ [e_\alpha, e_\sigma] &= -3e_\sigma, & [e_\alpha, e_\rho] &= -2e_\rho, & [e_\alpha, e_\zeta] &= -e_\zeta, \\ [e_\alpha, e_{\bar{\zeta}}] &= -e_{\bar{\zeta}}, & [e_\rho, e_\zeta] &= -e_\sigma, & [e_\rho, e_{\bar{\zeta}}] &= -e_\sigma, \\ [e_\zeta, e_{\bar{\zeta}}] &= -ie_\rho, \end{aligned}$$

the remaining brackets being equal to zero.

We refer to [5], p. 127-128, for the definition of a Cartan connection. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the subalgebra spanned by e_α , \mathfrak{G} the connected, simply connected Lie group whose Lie algebra is \mathfrak{g} and \mathfrak{G}_0 the closed 1-dimensional subgroup of \mathfrak{G} generated by \mathfrak{g}_0 . We notice that $\mathfrak{G}_0 \cong G_5$, so that P^5 is a principal bundle over M with structure group \mathfrak{G}_0 , and that $\dim \mathfrak{G}/\mathfrak{G}_0 = \dim M = 5$.

Let $(\Lambda, \tau, \sigma, \rho, \zeta, \bar{\zeta})$ be the coframe of 1-forms on P^5 whose structure equation are given by (2) – (3) and ω the 1-form on P with values in \mathfrak{g} defined by:

$$\omega(X) := \Lambda(X)e_\alpha + \tau(X)e_\tau + \sigma(X)e_\sigma + \rho(X)e_\rho + \zeta(X)e_\zeta + \bar{\zeta}(X)e_{\bar{\zeta}},$$

for $X \in T_p P^5$. We have:

Theorem 3. ω is a Cartan connection on P^5 .

Proof. We shall check that the following three conditions hold:

- (1) $\omega(e_\alpha^*) = e_\alpha$, where e_α^* is the vertical vector field on P^4 generated by the action of e_α ,
- (2) $R_a^* \omega = \text{Ad}(a^{-1}) \omega$ for every $a \in \mathfrak{G}_0$,
- (3) for each $p \in P^5$, ω_p is an isomorphism $T_p P^5 \xrightarrow{\omega_p} \mathfrak{g}$.

Condition (3) is trivially satisfied as $(\Lambda, \tau, \sigma, \rho, \zeta, \bar{\zeta})$ is a coframe on P^5 and thus defines a basis of $T_p^* P^5$ at each point p .

Condition (1) follows simply from the fact that Λ is a modified-Maurer Cartan form on P^5 :

$$\Lambda := \frac{da}{a} - X_\tau \tau - X_\sigma \sigma - X_\rho \rho - X_\zeta \zeta - X_{\bar{\zeta}} \bar{\zeta},$$

so that

$$\omega(e_\alpha^*) = \Lambda(e_{\alpha^*}) = e_\alpha,$$

as

$$\tau(e_\alpha) = \sigma(e_\alpha^*) = \rho(e_\alpha^*) = \zeta(e_\alpha^*) = \bar{\zeta}(e_\alpha^*) = 0, \quad \frac{da}{a}(e_\alpha^*) = 1,$$

since e_α^* is a vertical vector field on P^5 .

Condition (2) is equivalent to its infinitesimal counterpart:

$$\mathcal{L}_{e_\alpha^*} \omega = -\text{ad}_{e_\alpha} \omega,$$

where $\mathcal{L}_{e_\alpha^*} \omega$ is the Lie derivative of ω by the vector field e_α^* and where ad_{e_α} is the linear map $\mathfrak{g} \rightarrow \mathfrak{g}$ defined by: $\text{ad}_{e_\alpha}(X) = [e_\alpha, X]$. We determine $\mathcal{L}_{e_\alpha^*} \omega$ with the help of Cartan's formula:

$$\mathcal{L}_{e_\alpha^*} \omega = e_{\alpha^*} \lrcorner d\omega + d(e_\alpha^* \lrcorner \omega),$$

with

$$d(e_\alpha^* \lrcorner \omega) = 0$$

from condition (1). The structure equations (2)–(3) give:

$$e_{\alpha^*} \lrcorner d\omega = \begin{pmatrix} 0 \\ 4\tau \\ 3\sigma \\ 2\rho \\ \zeta \\ \bar{\zeta} \end{pmatrix},$$

which is easily seen being equal to $-\text{ad}_{e_\alpha} \omega$ from the Lie bracket structure of \mathfrak{g} . \square

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